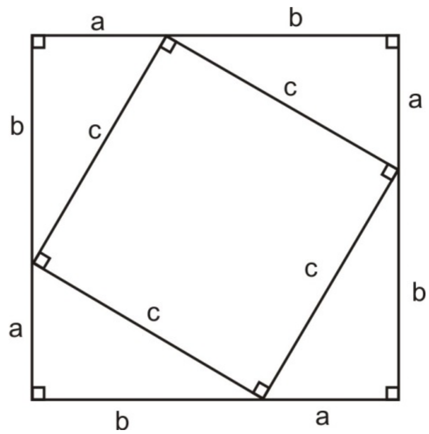
Important Proofs

Section 1.2 Pythagorean Theorem

Informal Proof of the Pythagorean Theorem:



One method of proving this theorem, given a right triangle with legs of a and b and hypotenuse c, is to make four copies of the triangle and arrange them so that their legs form an outer square and their hypotenuses form an inner square, as shown to the left.

The area of the outer square minus the area of the four triangles equals the area of the inner square, so:

$$(a+b)^2 - 4\left(\frac{1}{2}ab\right) = c^2$$

Simplifying gives us the Pythagorean Theorem:

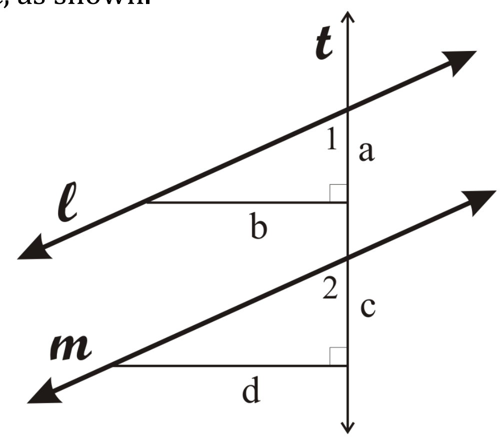
$$a^2 + 2ab + b^2 - 2ab = c^2$$
$$a^2 + b^2 = c^2$$

Section 2.2 Parallel and Perpendicular Lines

Informal Proof that parallel lines have the same slope

In the diagram below, we are given $\ell \parallel m$. A vertical transversal, *t*, is drawn, and horizontal segments *b* and *d* are drawn forming right angles with *t*, as shown.

$\angle 1 \cong \angle 2$ since they are corresponding angles formed by parallel lines intersected by a transversal. Therefore, the two triangles are similar by AA. Corresponding sides of similar triangles are in proportion, so $\frac{a}{b} = \frac{c}{d}$.

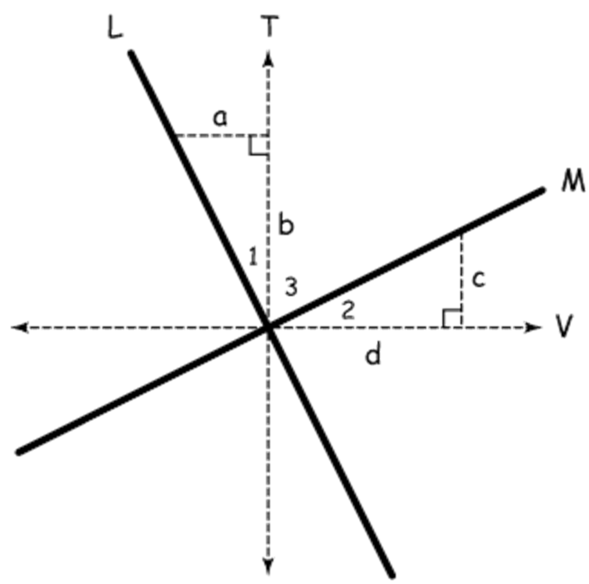


The slope of a line may be expressed as $\frac{rise}{run}$. So, the slope of $\ell = \frac{a}{b}$ and the slope of $m = \frac{c}{d}$. Therefore, the slopes are the two lines are equal.

Informal Proof that perpendicular lines have slopes that are opposite reciprocals

In the diagram below, we are given $L \perp M$. Through the point of intersection of *L* and *M*, lines *T* and *V* are drawn such that *T* is parallel to the *y*-axis and *V* is parallel to the *x*-axis (ie, $T \perp V$).

Right triangles are drawn as shown, with *b* = *d*. Since $m\angle 1 = 90° - m\angle 3$ and $m\angle 2 = 90° - m\angle 3$, we know that $m\angle 1 = m\angle 2$ (and therefore, $\angle 1 \cong \angle 2$). So, the two triangles are congruent by ASA. Therefore, *a* = *c*.



The slope of *L*, represented as $\frac{rise}{run}$, is $-\frac{b}{a}$.

The slope of *M*, represented as $\frac{rise}{run}$, is $\frac{c}{d}$.

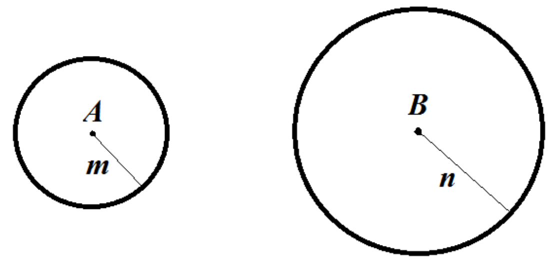
By substituting *a* for *c* and *b* for *d*, the slope of *M* may be rewritten as $\frac{a}{b}$. Therefore, the slopes of *L* and *M* are opposite reciprocals.

Section 6.4 Prove Congruency or Similarity

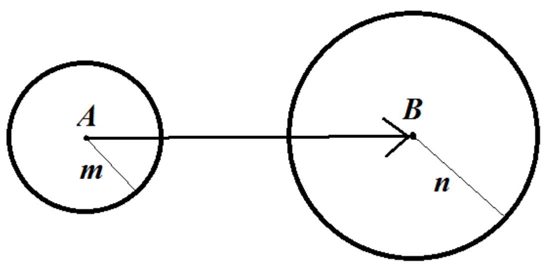
Informal proof that all circles are similar
All circles have the same shape, and we can prove they are similar by transformations.

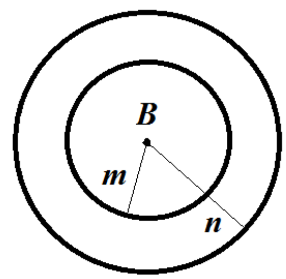
Consider any two circles, A and B, shown to the right, where circle A has a radius of m and circle B has a radius of n and $n \geq m$.

We can prove their similarity by showing a sequence of transformations that maps circle A to circle B.



First, translate circle A so that point A maps onto point B. The circles now have the same center.





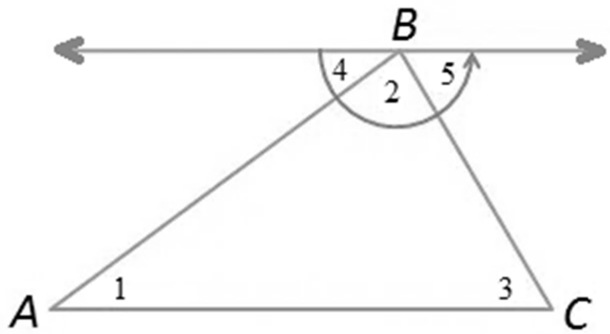
We can then dilate circle A by a scale factor of $\frac{n}{m}$ from its center. A radius of m, when scaled by a factor of $\frac{n}{m}$, will result in a radius of $m \cdot \frac{n}{m} = n$. Therefore, circle A maps onto circle B by a sequence of transformations – a translation and dilation – to prove that the circles are similar.

SECTION 9.1 ANGLES OF TRIANGLES

Proof of the Triangle Sum Theorem:

Given: $\triangle ABC$

Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 180°$



To prove this, we'll draw a line parallel to $\overline{AC}$ through vertex B.

Statements	*Reasons*
Draw a line parallel to $\overline{AC}$ through B	Parallel Postulate
$m\angle 4 + m\angle 2 + m\angle 5 = 180°$	Angles on one side of a straight line add to 180°.
$m\angle 1 = m\angle 4$ and $m\angle 3 = m\angle 5$	Alternate Interior Angles Theorem
$m\angle 1 + m\angle 2 + m\angle 3 = 180°$	Substitution

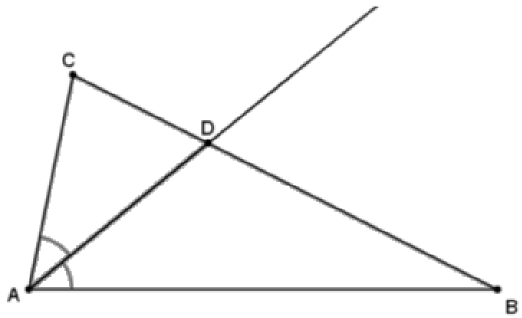
SECTION 10.4 TRIANGLE ANGLE BISECTOR THEOREM

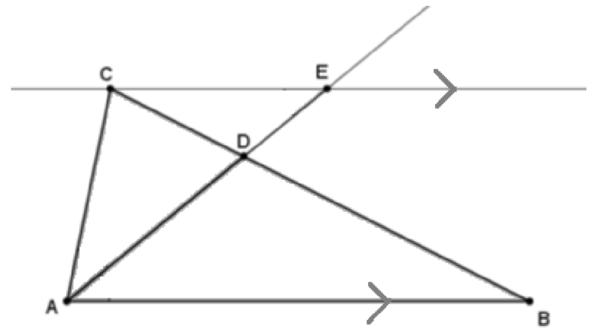
Proof of the Triangle Angle Bisector Theorem:

Given: $\triangle ABC$, $\overline{AD}$ bisects $\angle A$

Prove: $\frac{BD}{CD} = \frac{BA}{CA}$

To prove this, we'll extend $\overleftrightarrow{AD}$ and draw $\overleftrightarrow{CE} \parallel \overline{AB}$, where E is the point of intersection with $\overleftrightarrow{AD}$, and prove $\triangle BAD \sim \triangle CED$.





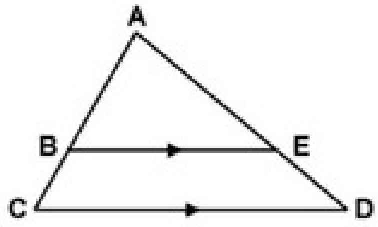
Statements	*Reasons*
$\overline{AD}$ bisects $\angle A$	Given
$\overline{CE} \parallel \overline{AB}$	Through a point not on a given line, a parallel line may be drawn
$\angle ECD \cong \angle ABD$ (A) $\angle CED \cong \angle BAD$ (A)	If a transversal intersects two parallel lines, alternate interior angles are congruent
$\triangle BAD \sim \triangle CED$	AA~
$\frac{BD}{CD} = \frac{BA}{CE}$	Side Proportionality
$\angle BAD \cong \angle CAD$	Definition of angle bisector
$\angle CED \cong \angle CAD$	Transitive Property
$CE = CA$	If two angles of a triangle are congruent, the opposite sides are congruent
$\frac{BD}{CD} = \frac{BA}{CA}$	Substitution Property

Section 10.5 Side Splitter Theorem

Proof of the Side Splitter Theorem:

Given: $\triangle ACD, \overleftrightarrow{ABC}, \overleftrightarrow{AED}, \overline{BE} \parallel \overline{CD}$.

Prove: $\frac{AB}{BC} = \frac{AE}{ED}$.



We can prove this theorem by first showing that $\triangle ABE \sim \triangle ACD$.

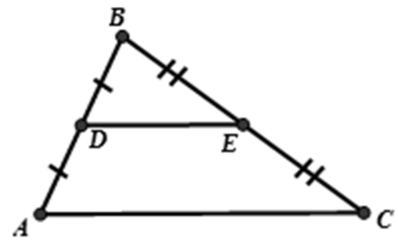
Statements		*Reasons*
$\overline{BE} \parallel \overline{CD}$		Given
$\angle A \cong \angle A$	(A)	Reflexive
$\angle ABE \cong \angle ACD$ *(or,* $\angle AEB \cong \angle ADC$*)*	(A)	If a transversal intersects two parallel lines, corresponding angles are congruent.
$\triangle ABE \sim \triangle ACD$		AA~
$\frac{AB}{AC} = \frac{AE}{AD}$		Side Proportionality
$AC = AB + BC$ $AD = AE + ED$		Segment Addition Postulate
$\frac{AB}{AB + BC} = \frac{AE}{AE + ED}$		Substitution
$AB(AE + ED) = AE(AB + BC)$		In a proportion, the product of the means = the product of the extremes
$AB \cdot AE + AB \cdot ED = AE \cdot AB + AE \cdot BC$		Distributive Property
$AB \cdot AE = AE \cdot AB$		Commutative Property
$AB \cdot ED = AE \cdot BC$		Subtraction Property
$\frac{AB}{BC} = \frac{AE}{ED}$		In a proportion, the product of the means = the product of the extremes

Section 10.6 Triangle Midsegment Theorem

Proof of the Triangle Midsegment Theorem:.

Given: $\triangle ABC$, D and E are the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively

Prove: $\overline{DE} \parallel \overline{AC}$ and $DE = \frac{1}{2}AC$

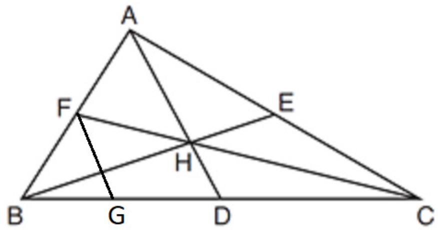


We can prove this theorem by first showing that $\triangle ABC \sim \triangle DBE$.

Statements		*Reasons*
D is the midpoint of $\overline{AB}$ E is the midpoint of $\overline{BC}$		Given
$\overline{DB} \cong \overline{AD}$, $\overline{EB} \cong \overline{CE}$		Definition of midpoint
$DB = AD$, $EB = CE$		Definition of congruence
$DB + AD = AB$		Segment Addition
$DB + DB = AB$, or $2DB = AB$		Substitution, and simplify
$\frac{AB}{DB} = 2$	(S)	Division
$\angle B \cong \angle B$	(A)	Reflexive
$EB + CE = CB$		Segment Addition
$EB + EB = CB$, or $2EB = CB$		Substitution, and simplify
$\frac{CB}{EB} = 2$	(S)	Division
$\triangle ABC \sim \triangle DBE$		SAS~
$\angle BDE \cong \angle BAC$		Corresponding angles of similar triangles are congruent
$\overline{DE} \parallel \overline{AC}$		If a transversal intersecting two lines forms congruent corresponding angles, then the lines are parallel
$\frac{AC}{DE} = 2$		Side Proportionality
$DE = \frac{1}{2}AC$		Solve for DE

SECTION 11.2 ORTHOCENTER AND CENTROID

Proof of the Centroid Median Theorem:
Given $\triangle ABC$ with centroid H below, we want to prove that $CH = 2 \cdot HF$.
(Proofs for the other two medians would be derived in the same way.)

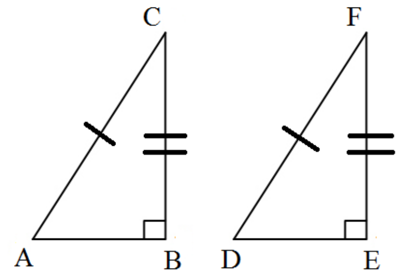


Statements	*Reasons*
$\triangle ABC$ with medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ drawn through centroid H	Given
D, E, and F are the midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively	Definition of median
Draw $\overline{FG} \parallel \overline{AD}$	Parallel Postulate
$\angle BFG \cong \angle BAD, \angle BGF \cong \angle BDA$	Corresponding Angles Theorem
$\triangle BFG \sim \triangle BAD$	AA~
$AF = \frac{1}{2}AB$	Definition of midpoint
$DG = \frac{1}{2}DB$	Side Splitter Theorem (in $\triangle ABD$)
$DB = CD$	Definition of midpoint
$DG = \frac{1}{2}CD$	Substitution
$\angle CHD \cong \angle CFG, \angle CDH \cong \angle CGF$	Corresponding Angles Theorem
$\triangle CHD \sim \triangle CFG$	AA~
$HF = \frac{1}{2}CH$	Side Splitter Theorem (in $\triangle CFG$)
$CH = 2 \cdot HF$	Multiplication Property

Proof of the Hypotenuse-Leg Theorem

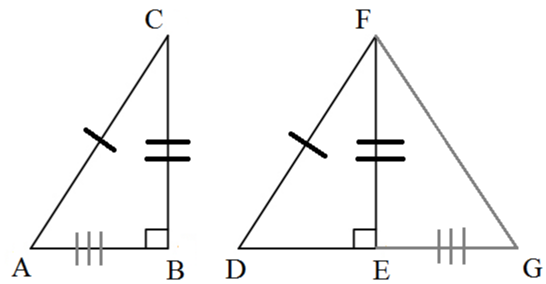
Given: Right triangles ABC and DEF,
Hypotenuses $\overline{AC} \cong \overline{DF}$, and legs $\overline{BC} \cong \overline{EF}$

Prove: $\triangle ABC \cong \triangle DEF$



To prove the triangles congruent, extend $\overline{DE}$ to point G such that $\overline{AB} \cong \overline{EG}$ and draw $\overline{FG}$.

Below, we prove $\triangle ABC \cong \triangle GEF$ (by SAS), then we prove $\triangle GEF \cong \triangle DEF$ (by AAS), which gives us $\triangle ABC \cong \triangle DEF$ by the Transitive Property.



Statements		*Reasons*
$\angle B$ and $\angle E$ are right angles, $\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$	(S)	Given
$\overline{FE} \perp \overline{DG}$		Definition of perpendicular
$\angle FEG$ is a right angle		Definition of perpendicular
$\angle ABC \cong \angle GEF$	(A)	All right angles are congruent
$\overline{AB} \cong \overline{EG}$	(S)	A line may be extended indefinitely or by any length
$\triangle ABC \cong \triangle GEF$		SAS
$\overline{AC} \cong \overline{FG}$		CPCTC
$\overline{DF} \cong \overline{FG}$	(S)	Transitive Property
$\triangle DFG$ is isosceles		Definition of isosceles
$\angle D \cong \angle G$	(A)	Base angles of isosceles triangles are congruent
$\angle DEF \cong \angle GEF$	(A)	All right angles are congruent
$\triangle GEF \cong \triangle DEF$		AAS
$\triangle ABC \cong \triangle DEF$		Transitive Property

SECTION 14.2 PROPERTIES OF QUADRILATERALS

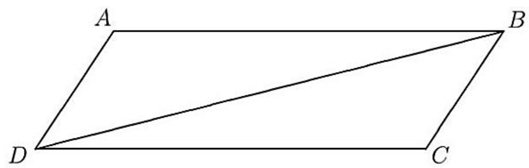
Proof that opposite sides and angles of a parallelogram are congruent

In this proof, we will first show that a diagonal divides the figure into two congruent triangles.

Given: In $\square ABCD$, diagonal $\overline{BD}$ is drawn.

Prove: $\overline{AD} \cong \overline{BC}$ and $\overline{AB} \cong \overline{DC}$

$\angle A \cong \angle C$ and $\angle ADC \cong \angle ABC$

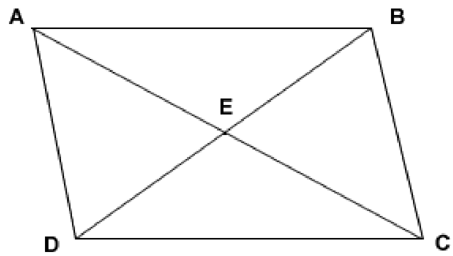


Statements		*Reasons*
$\square ABCD$		Given
$\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$		Definition of parallelogram
$\angle ADB \cong \angle CBD$ and $\angle ABD \cong \angle CDB$	(A) (A)	If a transversal intersects two parallel lines, alternate interior angles are congruent
$m\angle ADB = m\angle CBD$ and $m\angle ABD = m\angle CDB$		Definition of congruence
$m\angle ADB + m\angle CDB = m\angle ABD + m\angle CBD$		Addition
$m\angle ADC = m\angle ADB + m\angle CDB$ and $m\angle ABC = m\angle ABD + m\angle CBD$		Partition Postulate
$m\angle ADC = m\angle ABC$		Substitution
$\angle ADC \cong \angle ABC$		Definition of congruence
$\overline{BD} \cong \overline{BD}$	(S)	Reflexive
$\triangle ABD \cong \triangle CDB$		ASA
$\overline{AD} \cong \overline{BC}$ and $\overline{AB} \cong \overline{DC}$		CPCTC
$\angle A \cong \angle C$		CPCTC

Proof that the diagonals of a parallelogram bisect each other

Given: In $\square ABCD$, diagonals $\overline{BD}$ and $\overline{AC}$ are drawn, intersecting at E

Prove: $\overline{BD}$ bisects $\overline{AC}$ and $\overline{AC}$ bisects $\overline{BD}$

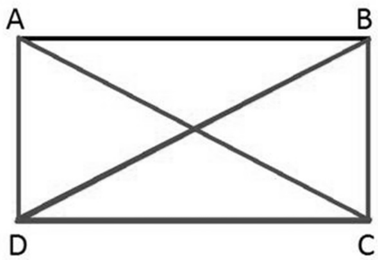


Statements		*Reasons*
$\square ABCD$		Given
$\angle BAE \cong \angle DCE$ and $\angle ABE \cong \angle CDE$	(A) (A)	If a transversal intersects two parallel lines, alternate interior angles are congruent
$\overline{AB} \cong \overline{CD}$	(S)	Opposite sides of a parallelogram are congruent
$\triangle ABE \cong \triangle CDE$		ASA
$\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$		CPCTC
$\overline{BD}$ bisects $\overline{AC}$ and $\overline{AC}$ bisects $\overline{BD}$		Definition of segment bisector

SECTION 14.4 PROVE TYPES OF QUADRILATERALS

Proof that a parallelogram with congruent diagonals is a rectangle

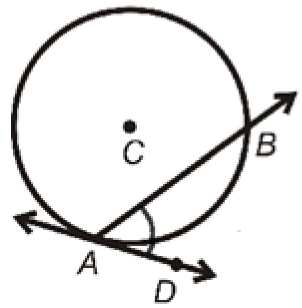
Given: In ▱$ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ are congruent
Prove: ▱$ABCD$ is a rectangle



Statements		*Reasons*
▱$ABCD$		Given
$\overline{AC} \cong \overline{BD}$	(S)	Given
$\overline{AD} \cong \overline{BC}$	(S)	Opposite sides of a parallelogram are congruent
$\overline{DC} \cong \overline{DC}$	(S)	Reflexive
$\triangle ADC \cong \triangle BCD$		SSS
$\angle ADC$ and $\angle BCD$ are supplementary		Consecutive angles of a parallelogram are supplementary
$\angle ADC \cong \angle BCD$		CPCTC
$\angle ADC$ and $\angle BCD$ are right angles		If supplementary angles are congruent, then they are right angles
$\angle ADC \cong \angle ABC$ and $\angle BCD \cong \angle BAD$		Opposite angles of a parallelogram are congruent
$\angle ABC$ and $\angle BAD$ are right angles		If an angle is congruent to a right angle, then it is a right angle
▱$ABCD$ is a rectangle		A rectangle is a parallelogram with four right angles (definition of rectangle)

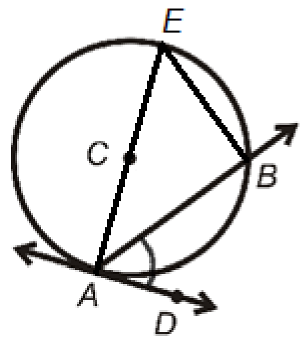
Section 15.4 Tangents

Proof of the Tangent-Chord Theorem:



Given: Tangent $\overleftrightarrow{AD}$ and chord $\overline{AB}$ intersect on the circle at A, forming $\angle BAD$.

Prove: $m\angle BAD = \frac{1}{2} m\,\widehat{AB}$

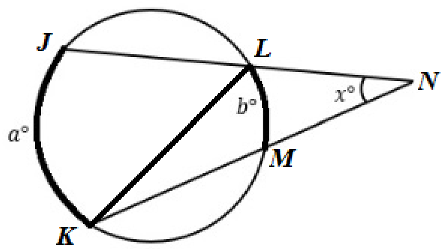


To prove this theorem, draw diameter $\overline{ACE}$ and chord $\overline{EB}$ to form
$\triangle ABE$, as shown below.

Statements	*Reasons*
Tangent $\overleftrightarrow{AD}$ and chord $\overline{AB}$ intersect on the circle at A. Diameter $\overline{ACE}$ and chord $\overline{EB}$ are drawn to form $\triangle ABE$.	Given
$\angle DAE$ is a right angle	Tangent-Radius Theorem
$\angle ABE$ is a right angle	the inscribed angle of a semicircle is a right angle
$\triangle ABE$ is a right triangle	definition of right triangle
$m\angle BAD = 90° - m\angle BAE$	adjacent angles that form a right angle are complementary
$m\angle AEB = 90° - m\angle BAE$	acute angles of a right triangle are complementary
$m\angle BAD = m\angle AEB$	Substitution
$m\angle AEB = \frac{1}{2} m\,\widehat{AB}$	an inscribed angle is one-half the measure of its intercepting arc
$m\angle BAD = \frac{1}{2} m\,\widehat{AB}$	substitution

SECTION 15.5 SECANTS

Informal Proof of Intersecting Secant Angles Theorem:
Given secants $\overline{JLN}$ and $\overline{KMN}$, draw $\overline{KL}$ to form $\triangle KLN$. $\text{m}\widehat{JK} = a$, $\text{m}\widehat{LM} = b$, and $\text{m}\angle N = x$.



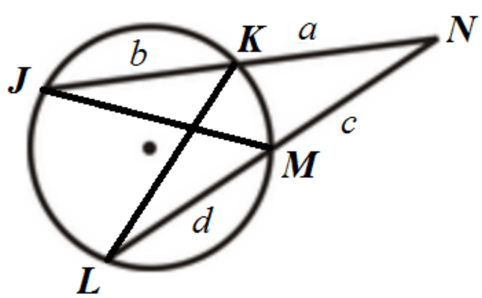
$\text{m}\angle JLK = \text{m}\angle K + \text{m}\angle N$ (exterior angle of a triangle)

$\text{m}\angle JLK = \frac{a}{2}$ and $\text{m}\angle K = \frac{b}{2}$ (inscribed angles)

$\frac{a}{2} = \frac{b}{2} + x$ (substitution)

$x = \frac{a-b}{2}$ (solve for x)

Informal Proof of Intersecting Secants Theorem:
This can be proven by drawing chords connecting the points of intersection of the secants with the circle, as shown below.



Since $\angle J \cong \angle L$ (inscribed angles of the same arc) and $\angle N \cong \angle N$ (reflexive), $\triangle NJM \sim \triangle NLK$ (AA~).

So, $\frac{a}{c+d} = \frac{c}{a+b}$, which gives us $a(a+b) = c(c+d)$.